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A note on pole placement by static output feedback for single-input systems

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Abstract: In this note we derive necessary and sufficient conditions for the solvability of the problem of pole placement by static output feedback formulated for single-input systems.

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Introduction

Consider the linear system Σ given by

$$\dot{x}(t) = Ax(t) + bu(t),$$
$$y(t) = Cx(t)$$

with state $x(t) \in \mathbb{R}^n$, control input $u(t) \in \mathbb{R}$ and measurement output $y(t) \in \mathbb{R}^p$. A, b and C are real matrices of dimensions $n \times n$, $n \times 1$ and $p \times n$, respectively. Assume that the linear system Σ is controlled by a linear static output feedback

$$u(t) = ky(t)$$

with k a real $1 \times p$ matrix. The resulting closed-loop system Σ_{cl} is described by

$$\dot{x}(t) = (A + bkC)x(t).$$

The poles of the closed-loop system Σ_{cl} are the eigenvalues of the matrix $A + bkC$. It is the purpose of the present note to investigate at which locations the poles of the closed-loop system Σ_{cl} can be placed using static output feedback.

To this end we assume that the linear system Σ is controllable, i.e. we assume that $q = n$, where $q = \text{rank}\ [b, Ab, \ldots, A^{n-1}b]$ (cf. [3]). Because, if the system Σ is not controllable, i.e. if $q < n$, it is possible to find a state-space transformation such that the system Σ can be partitioned as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

with A_{11}, A_{12}, A_{22}, b_1, C_1 and C_2 real matrices of dimensions $q \times q$, $q \times (n-q)$, $(n-q) \times (n-q)$, $q \times 1$, $p \times q$ and $p \times (n-q)$, respectively, and the pair (A_{11}, b_1) controllable (cf. [3]). With respect to this partitioning the closed-loop system Σ_{cl} obtained by the application of the static output feedback $u(t) = ky(t)$ is given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} + b_1kC_1 & A_{12} + b_1kC_2 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Hence, the poles of the closed-loop system Σ_{cl} consist of the eigenvalues of $A_{11} + b_1kC_1$ and the eigenvalues of A_{22}. The eigenvalues of A_{22} are known in advance and can not be shifted by static output feedback. From this reasoning it is clear that for the investigation at which locations the poles of the closed-loop system can be placed using static output feedback we may focus on systems Σ that are controllable.

Results

Letting im denote the image and ker the kernel of a matrix, we can formulate the following theorem which is the main result of this note.

Theorem. *Let Σ be a controllable system as described above and let $p(s)$ be a real monic polynomial of degree n. Then there exists a real $1 \times p$ matrix k such that*

$$p(s) = \det(sI - (A + bkC))$$

if and only if

$$p(A)\ker C \subset \operatorname{im}[b, Ab, \ldots, A^{n-2}b].$$

Proof. (only if). From the Cayley–Hamilton theorem it is clear that $p(A+bkC)=0$. If $x \in \ker C$, it follows by induction that for every i, $2 \leqslant i \leqslant n$, there exist

$$w_i \in \operatorname{im}[b, Ab, \ldots, A^{i-2}b]$$

such that

$$(A+bkC)^i x = A^i x + w_i.$$

Consequently, for every $x \in \ker C$ there exists a vector

$$w \in \operatorname{im}[b, Ab, \ldots, A^{n-2}b]$$

such that

$$0 = p(A+bkC)x = p(A)x + w.$$

Hence,

$$p(A)\ker C \subset \operatorname{im}[b, Ab, \ldots, A^{n-2}b].$$

(if). Since the pair (A, b) is controllable there exists a real (uniquely determined) $1 \times n$ matrix f such that

$$p(s) = \det(sI - (A+bf))$$

(cf. [5]). From the Ackermann formula (cf. [1,2]) it follows that

$$f = e_n^{\mathrm{T}}[b, Ab, \ldots, A^{n-1}b]^{-1}p(A)$$

where $^{\mathrm{T}}$ denotes transposition and e_i denotes the i-th unit vector in $\mathbb{R}^n$. Now there exists a real $1 \times p$ matrix k such that $f = kC$ if and only if $\ker C \subset \ker f$ (cf. [4]). From this it follows that there exists a $1 \times p$ matrix k such that

$$p(s) = \det(sI - (A+bkC))$$

if and only if

$$\ker C \subset \ker e_n^{\mathrm{T}}[b, Ab, \ldots, A^{n-1}b]^{-1}p(A).$$

In turn, the latter is equivalent to

$$p(A)\ker C \subset \ker e_n^{\mathrm{T}}[b, Ab, \ldots, A^{n-1}b]^{-1}.$$

The proof of the theorem is now completed by the observation that

$$\ker e_n^{\mathrm{T}}[b, Ab, \ldots, A^{n-1}b]^{-1} = \operatorname{im}[b, Ab, \ldots, A^{n-2}b]. \quad \square$$

Using the conditions of the above theorem we can investigate the existence of a real $1 \times p$ matrix k satisfying

$$p(s) = \det(sI - (A+bkC)),$$

whereupon k can be computed in a way as described in the proof of the (if) part. By dual reasoning a statement about pole placement by static output feedback for systems with single output and, possibly, multiple input can be derived. We omit this result and we continue with a special case of our pole-placement problem in which we assume that also the output is a scalar. The system, denoted Σ', is then given by

$$\dot{x}(t) = Ax(t) + bu(t),$$
$$y(t) = cx(t)$$

where $x(t)$, $u(t)$, A and b are as described before, $y(t) \in \mathbb{R}$ and c is a real $1 \times n$ matrix. For $i = 1, 2, \ldots, n$ let R_i and O_i be real matrices defined as

$$R_i = [b, Ab, \ldots, A^{i-1}b] \quad \text{and} \quad O_i = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{i-1} \end{bmatrix}.$$

We assume that the single-input/single-output system Σ' is minimal, i.e. rank R_n = rank $O_n = n$. The following result is a special case of our main theorem and states exactly when there is a scalar static output feedback such that the poles of the closed-loop system are at prescribed locations given by the zeros of the polynomial $p(s)$.

Corollary. *Let Σ' be a minimal system as given above and let $p(s)$ be a real polynomial of degree n. Then there exists a real number k such that*

$$p(s) = \det(sI - (A+bkc))$$

if and only if there exists an integer t, $1 \leqslant t < n$, such that

$$p(A)\ker O_{n-t} \subset \operatorname{im} R_t.$$

Furthermore, if such an integer exists then the latter subspace inclusion is valid for all integers t, $1 \leqslant t < n$.

Proof. From the previous theorem and its omitted dual version it is clear that the following statements are equivalent.

(a) There is a $k \in \mathbb{R}$ such that

$$p(s) = \det(sI - (A + bkc)).$$

(b) $p(A)\ker O_1 \subset \operatorname{im} R_{n-1}$.

(c) $p(A)\ker O_{n-1} \subset \operatorname{im} R_1$.

Without loss of generality we may assume that (cf. [3])

$$A = \begin{bmatrix} 0 & 0 & \dots & & & -r_0 \\ 1 & 0 & & & & -r_1 \\ 0 & 1 & & & & -r_2 \\ \vdots & & \ddots & & & \\ 0 & & & 1 & 0 & \vdots \\ 0 & 0 & \dots & 0 & 1 & -r_{n-1} \end{bmatrix}, \quad b = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ \\ b_{n-1} \end{bmatrix}$$

and $c = [0, 0, \dots, 0, 1]$.

Note that

$$\ker O_t = \operatorname{span}\{e_1, e_2, \dots, e_{n-t}\}$$

and $e_t = A^{t-1}e_1$ for all t, $1 \leqslant t < n$. Therefore

$$p(A)\ker O_t = p(A)\operatorname{span}\{e_1, Ae_1, \dots, A^{n-t-1}e_1\}$$
$$= \operatorname{span}\{h, Ah, \dots, A^{n-t-1}h\}$$

for all t, $1 \leqslant t < n$, where we have denoted $h = p(A)e_1$. Now suppose that

$$p(A)\ker O_t \subset \operatorname{im} R_{n-t}$$

for some t, $1 < t < n$. Then it follows that

$$\begin{aligned} p(A)\ker O_{t-1} &= \operatorname{span}\{h, Ah, \dots, A^{n-t}h\} \\ &= \operatorname{span}\{h, Ah, \dots, A^{n-t-1}h\} \\ &\quad + A\operatorname{span}\{h, Ah, \dots, A^{n-t-1}h\} \\ &= p(A)\ker O_t + Ap(A)\ker O_t \\ &\subset \operatorname{im} R_{n-t} + A \operatorname{im} R_{n-t} \\ &= \operatorname{im} R_{n-t+1}. \end{aligned}$$

Hence, for all t, $1 < t < n$, $p(A)\ker O_t \subset \operatorname{im} R_{n-t}$ implies

$$p(A)\ker O_{t-1} \subset \operatorname{im} R_{n-t+1}.$$

By the equivalence of the statements (a), (b) and (c) the proof of the corollary is now completed. □

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